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Coloured FRT algebra and its Yang–Baxterization leading to integrable models with non-additive R-matrices

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Abstract. A 'colour' representation of Faddeev–Reshetikhin–Takhtajan (FRT) algebra, which, in contrast to the standard case, is related to the coloured braid group representation with generic values of q is presented. Explicit realizations of $L^{(\pm)}$-matrices, occurring in this coloured variant of FRT algebra, are also obtained for the $U_q(gl(2))$ quantized algebra. Though these realizations are found to depend manifestly on the colour parameters, the underlying quantum group structure and associated co-product are interestingly free from such dependence. This allows us to perform the Yang–Baxterization of the coloured FRT algebra successfully, which leads to the construction of an ancestor Lax operator associated with a new non-additive-type quantum R-matrix. Through different realizations of this Lax operator, a new class of quantum integrable models representing 'colour' generalizations of the well known models, such as the lattice sine–Gordon model, the Ablowitz–Ladik model lattice and the derivative nonlinear Schrödinger model etc, is generated.

1. Introduction

An elegant approach to quantized algebra and quantum group structures, which has a close relation with the theory of integrable systems, was formulated by Faddeev, Reshetikhin and Takhtajan (FRT) [1] by exploiting the duality condition of the Hopf algebra. In this approach, the quantum-group related algebras are represented in matrix form and may be expressed as

$$R^+ L_1^{(\pm)} L_2^{(\pm)} L_2^{(\pm)} L_1^{(\pm)} R^+ \tag{1.1a}$$

$$R^+ L_1^{(+)} L_2^{(-)} L_2^{(-)} L_1^{(+)} R^+ . \tag{1.1b}$$

These relations are the characteristic equations of the FRT algebra, where $L_1^{(\pm)} = L^{(\pm)} \otimes \mathbf{1}$, $L_2^{(\pm)} = \mathbf{1} \otimes L^{(\pm)}$ and $L^{(\pm)}$ are upper (lower) triangular matrices with operator-valued elements which are related to the generators of the quantized algebra. Due to the associativity condition on (1.1), the matrix $R^+ \in M_{n^2}(C)$ satisfies the relation

$$R_{12}^+ R_{13}^+ R_{23}^+ = R_{23}^+ R_{13}^+ R_{12}^+ \tag{1.2}$$

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where R^+_{ij} acts non-trivially on the direct product space $C^n \otimes C^n \otimes C^n$, on the ith and jth spaces, and like an identity on the remaining space (e.g. $R^+_{12} \equiv R \otimes \mathbf{1}$, etc). Note that the R^+-matrix leads to a braid group representation (BGR) for $\hat{R}^+ = \mathcal{P}R^+$ ($\mathcal{P}$ being the permutation operator with the property $\mathcal{P}A \otimes B = B \otimes A\mathcal{P}$)

$$\hat{R}^+_{12}\hat{R}^+_{23}\hat{R}^+_{12} = \hat{R}^+_{23}\hat{R}^+_{12}\hat{R}^+_{23}.$$

However, we shall call the R^+-matrix itself the BGR for our convenience in what follows.

It is interesting to observe that the FRT relations (1.1) are very similar in form to the quantum Yang–Baxter equation (QYBE)

$$R(\lambda,\mu)L_1(\lambda)L_2(\mu) = L_2(\mu)L_1(\lambda)R(\lambda,\mu) \tag{1.3}$$

which plays a key role in the context of quantum integrable theory [2]. Here, λ and μ are spectral parameters (which may be multicomponent), $L(\lambda)$ is the Lax operator of the related integrable model and $R(\lambda,\mu)$ is the corresponding quantum R-matrix satisfying the Yang–Baxter equation (YBE)

$$R_{12}(\lambda,\mu)R_{13}(\lambda,\gamma)R_{23}(\mu,\gamma) = R_{23}(\mu,\gamma)R_{13}(\lambda,\gamma)R_{12}(\lambda,\mu). \tag{1.4}$$

However, in contrast to FRT algebra (1.1), QYBE (1.3) depends on the spectral parameters λ, μ and is represented by a single matrix relation. Thus, the natural question arises as to whether it is possible to 'Yang–Baxterize' the FRT algebra, i.e. to construct a spectral-parameter-dependent Lax operator $L(\lambda)$ and the corresponding $R(\lambda,\mu)$-matrix (which satisfies the QYBE) out of the elements contained in the FRT algebra. This possibility was actually realized [3] for the associated BGR, which, while satisfying the Hecke algebra, was also found to be useful in generating a wide class of quantum integrable models through different realizations of the FRT algebra [4]. Remarkably, in all models belonging to this class, the related quantum R-matrix is additive, i.e. $R(\lambda,\mu) \equiv R(\lambda-\mu)$.

On the other hand, in recent years, a different class of integrable model with more general non-additive-type R-matrix solutions of YBE (1.4) has been discovered [5–7]. Therefore, one may also ask whether Yang–Baxterization of the FRT algebra associated with the standard BGR, as developed in the additive case, can be pursued even for its non-additive generalization? Such a construction might finally lead to Lax operators of integrable models with non-additive R-matrix solutions.

It is encouraging to observe that possible generalizations of the BGR suitable for this purpose have already been described in recent literature and are sometimes called the 'coloured' braid group representations (CBGR) [8–11], which obey the relation

$$R_{12}^{+(\lambda,\mu)}R_{13}^{+(\lambda,\gamma)}R_{23}^{+(\mu,\gamma)} = R_{23}^{+(\mu,\gamma)}R_{13}^{+(\lambda,\gamma)}R_{12}^{+(\lambda,\mu)}. \tag{1.5}$$

Although $\hat{R}^{+(\lambda,\mu)} = \mathcal{P}R^{+(\lambda,\mu)}$ is usually defined as the CBGR, we shall the call $R^{+(\lambda,\mu)}$-matrix itself the CBGR in what follows, parallel to the additive case. Apparently, there can be different approaches to constructing such a CBGR. In [8, 9] an infinite-dimensional representation of $U_q(sl(2))$ was considered and the 'colour' index was introduced as the values of the corresponding Casimir operator. However, for obtaining finite-dimensional CBGRs, q has to be restricted to the root of unity. On the other hand, Burdik and Hellinger [10] have followed another original path where deformations of non-semisimple Lie algebras like $U_q(gl(2))$ were considered. Due to the splicing of $U_q(gl(2))$ into $U_q(sl(2))$ and

$Uu(1)$, an extra Casimir operator enters into the picture—the eigenvalues $\{\lambda\}$ of which, corresponding to the representation Π_λ^n, now serve as the colour degrees of freedom. This, in turn, gives a finite-dimensional CBGR even for the generic values of q. Finally, another approach, adopted in [12], may be mentioned where the CBGR (related to $U_q(gl(N))$) was obtained directly from the standard BGR in the 'particle-conserving' case by exploiting a symmetry transformation of the YBE. It may be noted here that if the colour indices $\{\lambda\}$ are interpreted as the spectral parameters (CBGR) themselves, as suggested by the form (1.5), then they may be considered to be the non-additive-type R-matrix solutions of the YBE (1.4). However, the CBGR is usually upper (lower) triangular in form, while the non-additive R-matrices, like their additive partners, could be of more general form. Therefore, we may ask whether, parallel to the construction of additive-type R-matrix solutions of the YBE from the BGR [13], one can also formulate a Yang–Baxterization scheme for the non-additive case starting from the CBGR. This point is successfully dealt with in section 2 where an explicit $(N^2 \times N^2)$ R-matrix with non-additive and multicomponent spectral-parameter dependence is constructed, starting from the CBGR with generic values of q. Another interesting problem is to explore whether the form of $L^{(\pm)}$-matrices, appearing in the FRT algebra (1.1), can be generalized properly to make them compatible with the CBGR (1.5). In section 3, we find such a generalized form of $L^{(\pm)}$, related indeed with the CBGR, which also yields the required co-product structures.

The more important question of physical relevance, as raised above, is whether the FRT algebra, related to the CBGR, can be Yang–Baxterized to give solutions of the QYBE (1.3) in analogy with the additive case. In section 4 we investigate this problem and successfully perform the Yang–Baxterization of the coloured FRT algebra in the $U_q(gl(N))$ case yielding the explicit form of an ancestor Lax operator associated with a new non-additive quantum R-matrix. Section 5 gives concrete realizations of the ancestor model, generating a new class of integrable models. Section 6 is the concluding section.

2. The construction of a non-additive R-matrix from the CBGR

As is well known, for a quasitriangular Hopf algebra $\mathcal{A}$ there exists an invertible universal $\mathcal{R}$-matrix ($\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$) such that it interrelates comultiplications Δ, Δ' through $\Delta(a)\mathcal{R} = \mathcal{R}\Delta'(a)$ (where $a \in \mathcal{A}$) and satisfies the following conditions

$$(\mathrm{id} \otimes \Delta)\mathcal{R}\mathcal{R}_{13}\mathcal{R}_{12} \qquad (\Delta \otimes \mathrm{id})\mathcal{R}\mathcal{R}_{13}\mathcal{R}_{23} \qquad (S \otimes \mathrm{id})\mathcal{R}\mathcal{R}^{-1}$$

where S is the antipode. The above relations also imply that the R-matrix would be a solution of the spectrally parameterless YBE (1.2).

If one now considers the case of $U_q(gl(2))$, where apart from the usual generators $S_3, S_\pm$ of $U_q(sl(2))$, a central element, or Casimir-like operator Λ, is included with the commutation relations [14]

$$\begin{aligned} &[S_3, S_\pm] = \pm S_\pm \qquad [S_+ S_-] = \frac{\sin(2\alpha S_3)}{\sin\alpha} \\ &[\Lambda, S_\pm] = [\Lambda, S_3] = 0 \qquad q = \mathrm{e}^{\mathrm{i}\alpha}. \end{aligned} \tag{2.1}$$

As a result the standard comultiplication is also modified to yield

$$\begin{aligned} &\Delta(S_+) = S_+ \otimes q^{-S_3} \cdot (qs)^\Lambda + (s/q)^\Lambda \cdot q^{S_3} \otimes S_+ \\ &\Delta(S_-) = S_- \otimes q^{-S_3} \cdot (qs)^{-\Lambda} + (s/q)^{-\Lambda} \cdot q^{S_3} \otimes S_- \\ &\Delta(S_3) = S_3 \otimes \mathbf{1} + \mathbf{1} \otimes S_3 \qquad \Delta(\Lambda) = \Lambda \otimes \mathbf{1} + \mathbf{1} \otimes \Lambda \end{aligned} \tag{2.2}$$

where an additional parameter s may appear due to the symmetry of the algebra. The other Hopf algebraic structures, such as co-unit and antipode, can be consistently defined and the universal $\mathcal{R}$-matrix may be constructed as [10]

$$\mathcal{R} = q^{2(S_3\otimes S_3+S_3\otimes\Lambda-\Lambda\otimes S_3)} \sum_{m=0}^{\infty} \frac{(1-q^{-2})^m}{[m,q^{-2}]!} (q^{S_3}(qs)^{-\Lambda}S_+)^m \otimes \left(q^{-S_3}\left(\frac{s}{q}\right)^{\Lambda} S_-\right)^m \tag{2.3}$$

where $[m,q]! = [m,q]\cdot[m-1,q]\ldots 1$ with $[m,q] = (1-q^m)/(1-q)$.

Denoting now the eigenvalue of the Casimir-like operator Λ by λ and the corresponding n-dimensional irreducible representation of algebra (2.1) by Π_λ^n, we may obtain the 'colour' representation $(\Pi_\lambda^n \otimes \Pi_\mu^n)\mathcal{R}$, which gives a finite-dimensional CBGR satisfying (1.4). In particular, for the two-dimensional representation Π_λ^2, one gets the CBGR [10] as

$$R^{+(\lambda,\mu)} \begin{pmatrix} q^{1-(\lambda-\mu)} & & & \\ & q^{\lambda+\mu} & (q-q^{-1})s^{-(\lambda-\mu)} & \\ & 0 & q^{-(\lambda+\mu)} & \\ & & & q^{1+(\lambda-\mu)} \end{pmatrix}. \tag{2.4}$$

This type of R-matrix solution was also obtained in [15].

In another recent development [12], similar CBGRs, related to the fundamental representation of $U_q(gl(N))$, were obtained directly from the standard BGR by using a symmetry transformation of the YBE. It has been shown that, if $R(\lambda,\mu)$ is a solution of the YBE (1.4) with the 'particle-conserving' constraint (i.e. its elements R_{ij}^{kl} are non-zero only when the 'incoming particles' (i,j) are some permutations of the outgoing ones (k,l)), then one may construct some more general solutions $\tilde{R}(\lambda,\lambda';\mu,\mu')$ depending on the two-component spectral parameters with elements given by

$$\tilde{R}_{ij}^{kl}(\lambda,\lambda';\mu,\mu') = R_{ij}^{kl}(\lambda,\mu)\frac{u_l^{(1)}(\lambda')u_j^{(2)}(\lambda')}{u_i^{(1)}(\mu')u_k^{(2)}(\mu')}. \tag{2.5}$$

Here, the indices i,j,k,l run from 1 to N and $u_i^{(1)}(\lambda')$, $u_i^{(2)}(\lambda')$ are $2N$ arbitrary spectral-parameter-dependent functions. This symmetry transformation of the YBE may be written in matrix form as a 'gauge transformation'

$$R(\lambda,\lambda';\mu,\mu') = F^{-1}(\lambda',\mu')R(\lambda,\mu)\tilde{F}^{-1}(\lambda',\mu') \tag{2.6}$$

where

$$F(\lambda',\mu') = \sum_{ij}\frac{u_i^{(1)}(\mu')}{u_j^{(2)}(\lambda')}e_{ii}\otimes e_{jj} \qquad \tilde{F}(\lambda',\mu') = \sum_{ij}\frac{u_i^{(2)}(\mu')}{u_j^{(1)}(\lambda')}e_{ii}\otimes e_{jj}$$

with e_{ij} the basis of $gl(N)$ with $(e_{ij})_{kl} = \delta_{ik}\delta_{jl}$. Consequently, starting from the standard BGR $R^{\pm}$ which is related to the fundamental representation of $U_q(sl(N))$ [16],

$$R^{\pm} = \sum_i q^{\pm 1}e_{ii}\otimes e_{ii} + \sum_{i\neq j}\phi_{ij}\cdot e_{ii}\otimes e_{jj} \pm (q-q^{-1})\sum_{\substack{i<j\\(i>j)}} e_{ij}\otimes e_{ji} \tag{2.7}$$

which evidently satisfies the 'particle-conserving' restriction and, using (2.6) again, one derives the corresponding CBGR as

$$R^{\pm(\lambda,\mu)} = F^{-1}(\lambda,\mu)\cdot R^{\pm}\cdot \tilde{F}^{-1}(\lambda,\mu)$$
$$= \sum_i q^{\pm 1}\frac{u_i^{(1)}(\lambda)u_i^{(2)}(\lambda)}{u_i^{(1)}(\mu)u_i^{(2)}(\mu)} e_{ii}\otimes e_{ii} + \sum_{i\neq j}\phi_{ij}\frac{u_j^{(1)}(\lambda)u_j^{(2)}(\lambda)}{u_i^{(1)}(\mu)u_i^{(2)}(\mu)} e_{ii}\otimes e_{jj}$$
$$\pm (q-q^{-1})\sum_{\substack{i<j\\(i>j)}}\frac{u_i^{(1)}(\lambda)u_j^{(2)}(\lambda)}{u_i^{(1)}(\mu)u_j^{(2)}(\mu)} e_{ij}\otimes e_{ji} \tag{2.8}$$

where ϕ_{ij} are arbitrary constants with the condition $\phi_{ij}\cdot\phi_{ji}=1$. Now, it is interesting to observe that in the particular case $N=2$, along with the choice

$$\phi_{12}=1 \qquad u_1^{(1)}(\lambda)=1 \qquad u_2^{(1)}(\lambda)=(qs)^{\lambda} \qquad u_1^{(2)}(\lambda)=q^{-\lambda} \qquad u_2^{(2)}(\lambda)=s^{-\lambda} \tag{2.9}$$

the form of the CBGR $R^{+(\lambda,\mu)}$ in (2.8) reduces exactly to the CBGR (2.4), which was obtained by Burdik *et al* from the universal $\mathcal{R}$-matrix related to $U_q(gl(2))$ in its fundamental representation. On the other hand, $R^{-(\lambda,\mu)}$ in (2.8), under the same condition (2.9), reduces to

$$R^{-(\lambda,\mu)} = \begin{pmatrix} q^{-(1+\lambda-\mu)} & & & \\ & q^{\lambda+\mu} & 0 & \\ & -(q-q^{-1})s^{\lambda-\mu} & q^{-(\lambda+\mu)} & \\ & & & q^{-1+(\lambda-\mu)} \end{pmatrix}. \tag{2.10}$$

We may hope that CBGR (2.8) with arbitrary N will also be similarly related to the fundamental representation of $U_q(gl(N))$.

In analogy with the Yang–Baxterization scheme in Jones [13] for the additive case, we would now like to construct the non-additive R-matrix solution of YBE (1.4) starting from the CBGR (2.8). Similar Yang–Baxterization was also considered by Murakami [11], but for the restrictive value of q as the root of unity ($q^2=-1$) and in the particular case $N=2$. For the BGR R^+ satisfying the extra Hecke condition

$$R^+ - R^- = (q-q^{-1})\cdot\mathcal{P} \qquad R^- = \mathcal{P}(R^+)^{-1}\mathcal{P} \tag{2.11}$$

where $\mathcal{P}$ is the permutation operator as defined before, Jones proposed the Yang–Baxterized form

$$R(\lambda'-\mu') = q^{\lambda'-\mu'}R^+ - q^{-(\lambda'-\mu')}R^- \tag{2.12}$$

yielding the additive R-matrix solution of the YBE. It may be noted that the Hecke condition (2.11) is satisfied by BGR (2.7) corresponding to $U_q(sl(N))$ in the fundamental representation and, therefore, Yang–Baxterization (2.12) is well applicable to this class of R-matrices.

Returning to CBGR (2.8), we find that in close analogy with the Yang–Baxterization (2.12) performed in the additive case, it is possible to construct a non-additive solution of YBE (1.4) in the form

$$R(\lambda,\lambda';\mu,\mu') = q^{(\lambda'-\mu')}\cdot R^{+(\lambda,\mu)} - q^{-(\lambda'-\mu')}\cdot R^{-(\lambda,\mu)} \tag{2.13}$$

which depends on two-component spectral parameters. This fact may easily be verified by using expression (2.8), which rewrites the right-hand side of (2.13) into

$$F^{-1}(\lambda,\mu)R(\lambda'-\mu')\tilde{F}^{-1}(\lambda,\mu) \tag{2.14}$$

where $R(\lambda'-\mu')$ is of the same form as in Yang–Baxterization (2.12) related to the additive case. Since $R^{\pm}$-matrices (2.7) are 'particle conserving', the $R(\lambda'-\mu')$-matrix, being a linear combination of R-matrices, is also a 'particle-conserving' solution of the YBE. Therefore, due to the symmetry transformation (2.6), expression (2.14) should be a new solution of YBE (1.4), thus proving the validity of Yang–Baxterization (2.13). It is interesting to note here that, in the particular case $N=2$ and with degenerate spectral parameters $\lambda=\lambda'$, $\mu=\mu'$, the R-matrix (2.13) coincides exactly with the asymmetric six-vertex solution corresponding to the two-dimensional statistical model of ferroelectrics in an external electric field with both horizontal and vertical components [7]. On the other hand, if we suppose that functions $u_i^{(a)}(\lambda)$ are independent of the spectral parameters, we may recover from (2.13), with (2.8), the R-matrix for the statistical model due to Perk and Schultz [17]. The explicit construction of the non-additive and multicomponent spectral-parameter-dependent $(N^2\times N^2)$ R-matrix solution (2.13) is our main result in this section. Therefore, to see its structure more closely we present the $N=2$ case for the particular choice (2.9) of functions $u_i^{(a)}(\lambda)$ related to the CBGRs (2.4) and (2.10)

$$R(\lambda,\lambda';\mu,\mu') = \begin{pmatrix} q^{-(\lambda-\mu)}a(\lambda'-\mu') & & & \\ & q^{(\lambda+\mu)}b(\lambda'-\mu') & s^{-(\lambda-\mu)}c_+(\lambda'-\mu') & \\ & s^{(\lambda-\mu)}c_-(\lambda'-\mu') & q^{-(\lambda+\mu)}b(\lambda'-\mu') & \\ & & & q^{(\lambda-\mu)}a(\lambda'-\mu') \end{pmatrix} \tag{2.15}$$

where $a(\lambda'-\mu')=\sin\alpha(\lambda'-\mu'+1)$, $b(\lambda'-\mu')=\sin\alpha(\lambda'-\mu')$, $c_\pm(\lambda'-\mu')=\sin\alpha q^{\pm(\lambda'-\mu')}$ and q is taken as $e^{i\alpha}$. It will be shown in section 4 that the same R-matrix (2.15) may be associated with a whole class of new integrable systems.

3. Coloured FRT algebra related to the CBGR

The FRT algebra may be given by the relations (1.1a), (1.1b), where the related R^+-matrix is the standard BGR satisfying (1.2). On the other hand, the CBGR $R^{+(\lambda,\mu)}$ satisfies a more general equation (1.5) and therefore the aim is to explore the possibility of constructing a more general form of $L^{(\pm)}$-matrices in the FRT algebra to make them compatible with the CBGR. Since, in the coloured case, the $L^{(+)}$-matrix may be seen as the specific representation of the universal $\mathcal{R}$-matrix in the form $L^{(+)}(\lambda)=(\Pi_\lambda^n\otimes\mathrm{id})\mathcal{R}$, it is expected to depend explicitly on the parameter λ and consequently the coloured FRT algebra should take the following form:

$$R^{+(\lambda,\mu)}L_1^{(\pm)}(\lambda)L_2^{(\pm)}(\mu)=L_2^{(\pm)}(\mu)L_1^{(\pm)}(\lambda)R^{+(\lambda,\mu)} \tag{3.1a}$$

$$R^{+(\lambda,\mu)}L_1^{(+)}(\lambda)L_2^{(-)}(\mu)=L_2^{(-)}(\mu)L_1^{(+)}(\lambda)R^{+(\lambda,\mu)} \tag{3.1b}$$

where $L_1^{(\pm)}(\lambda)=L^{(\pm)}(\lambda)\otimes\mathbf{1}$, $L_2^{(\pm)}(\lambda)=\mathbf{1}\otimes L^{(\pm)}(\lambda)$, ($L^{(\pm)}(\lambda)$ being upper (lower) triangular matrices) and $R^{+(\lambda,\mu)}$ is the related CBGR. One should note that a similar form for the FRT algebra was previously introduced by Faddeev *et al* [18] in connection with the Kac–Moody algebra. We hope that, similar to the standard FRT algebra, the above coloured version will hold in the general case. However, we show here its validity for CBGR (2.4) through explicit construction of the corresponding $L^{\pm}(\lambda)$-matrices.

For this purpose, for the $L^{(\pm)}$-matrices related to the standard BGR [3],

$$L^{(+)} = \begin{pmatrix} \tau_1^+ & \tau_{21} \\ 0 & \tau_2^+ \end{pmatrix} \qquad L^{(-)} = \begin{pmatrix} \tau_1^- & 0 \\ \tau_{12} & \tau_2^- \end{pmatrix}$$

we propose some colour generalizations by introducing the parameter λ in the form

$$L^{(+)}(\lambda) = G(\lambda) \begin{pmatrix} q^{\Lambda}\tau_1^+ & (qs)^{-\lambda} \cdot s^{\Lambda}\tau_{21} \\ 0 & q^{-\Lambda}\tau_2^+ \end{pmatrix} \tag{3.2a}$$

$$L^{(-)}(\lambda) = G(\lambda) \begin{pmatrix} q^{\Lambda}\tau_1^- & 0 \\ (qs)^{\lambda} \cdot s^{-\Lambda}\tau_{12} & q^{-\Lambda}\tau_2^- \end{pmatrix} \tag{3.2b}$$

where $G(\lambda) = (\tau_1^-\tau_2^+)^{\lambda}$; we expect that this specific choice of $L^{(\pm)}(\lambda)$ will lead to a quantized algebra which is independent of the colour parameters λ, μ for the as yet unspecified generators τ. Though the above forms of $L^{(\pm)}(\lambda)$ look rather complicated, one may observe that at $\lambda = \Lambda = 0$, they reduce simply to the known $L^{(\pm)}$-matrices related to the standard case. It may also be noted that the $L^{(\pm)}(\lambda)$-matrices in (3.2*a*), (3.2*b*) become similar to the $L^{(\pm)}$-matrices related to the r, q-deformed BGR which appeared in [16] if one fixes the colour parameter λ and interprets it as some function of the deformation parameter. Nevertheless, such $L^{(\pm)}$-matrices would only be the solutions of the standard FRT algebra (1.1*a*), (1.1*b*) and the deformation parameters would necessarily take some fixed values. In our case, however, the parameter λ, appearing in the coloured representation of the FRT algebra (3.1*a*), (3.1*b*), may vary arbitrarily describing a more general situation.

Inserting CBGR (2.4) and the explicit forms of $L^{(\pm)}(\lambda)$ (3.2*a*), (3.2*b*) in (3.1*a*), (3.1*b*), we find, interestingly, that the coloured FRT algebra reduces finally into the following algebraic relations for $\tau_i^{\pm}$, τ_{ij} $(i, j = 1, 2)$ and Λ, which are evidently free from colour parameters

$$\begin{aligned} &\tau_i^{\pm}\tau_{ij} = q^{\pm 1}\tau_{ij}\tau_i^{\pm} \qquad \tau_i^{\pm}\tau_{ji} = q^{\mp 1}\tau_{ji}\tau_i^{\pm} \\ &[\tau_{12}, \tau_{21}] = -(q - q^{-1})(\tau_1^+\tau_2^- - \tau_1^-\tau_2^+) \qquad [\tau_i^{\pm}, \Lambda] = [\tau_{ij}, \Lambda] = 0 \end{aligned} \tag{3.3}$$

with all $\tau_i^{\pm}$ commuting among themselves. It may be verified that, in addition to the central element Λ, there exist other Casimir operators of this algebra such as

$$\begin{aligned} &D_1 = \tau_1^+\tau_1^- \qquad D_2 = \tau_2^+\tau_2^- \qquad D_3 = \tau_1^+\tau_2^+ \\ &\tilde{D} = 2\cos\alpha(\tau_1^+\tau_2^- + \tau_1^-\tau_2^+) - [\tau_{12}, \tau_{21}]_+. \end{aligned} \tag{3.4}$$

We notice that, apart from the generator Λ, algebra (3.3) thus obtained reproduces the same extended trigonometric Sklyanin algebra as that found in the case of the standard BGR [3], while for the choice of generators τ in the particular form

$$\tau_1^{\pm} = q^{\pm S_3} \qquad \tau_2^{\pm} = q^{\mp S_3} \qquad \tau_{12} = -(q - q^{-1})S_+ \qquad \tau_{21} = (q - q^{-1})S_- \tag{3.5}$$

it reduces to the quantized algebra $U_q(gl(2))$ given as (2.1). Thus we see that the coloured algebra (3.1*a*), (3.1*b*), related to CBGR (2.4), is indeed able to generate the underlying quantum group structure which is independent of colour parameters λ, μ. Since the construction of $L^{(\pm)}(\lambda)$ given by (3.2*a*), (3.2*b*) is the main result of this section, we present it through more popular generators $S_\pm, S_3, \Lambda$ of the quantum algebra $U_q(gl(2))$ using (3.5) as

$$\begin{aligned} L^{(+)}(\lambda) &= q^{-2\lambda S_3}\begin{pmatrix} q^{\Lambda+S_3} & (q-q^{-1})(qs)^{-\lambda}\cdot s^{\Lambda}S_- \\ 0 & q^{-(\Lambda+S_3)} \end{pmatrix} \\ L^{(-)}(\lambda) &= q^{-2\lambda S_3}\begin{pmatrix} q^{\Lambda-S_3} & 0 \\ -(q-q^{-1})(qs)^{\lambda}\cdot s^{-\Lambda}S_+ & q^{-\Lambda+S_3} \end{pmatrix} \end{aligned} \tag{3.6}$$

which are fascinating to compare with the standard $L^\pm$ [1]. We may note further that, similar to the FRT algebra (1.1), its colour counterpart (3.1) also exhibits the symmetry that, if $L^{(\pm,1)}(\lambda)$ and $L^{(\pm,2)}(\lambda)$ are two independent solutions of the algebra acting on different quantum spaces, then their matrix product $\Delta L^{(\pm)}(\lambda) = (L^{(\pm,1)}(\lambda)\cdot L^{(\pm,2)}(\lambda))$ is also a solution with the same CBGR. Using this important property and the explicit forms (3.6) of $L^{(\pm)}(\lambda)$, we may derive the related co-product structure of the quantized algebra. Remarkably, one finds that, although the $L^{(\pm)}(\lambda)$-matrices contain the colour parameter λ in a complicated way, the resultant co-product for the generators $S_\pm$, S_3 and Λ is free from such parameters and, in fact, coincides with the known co-product (2.2) of $U_q(gl(2))$.

Similarly, it may be shown that, if we start from the CBGR $R^{-(\lambda,\mu)}$, given by (2.10), and the same form (3.2*a*), (3.2*b*) of $L^{(\pm)}(\lambda)$, the complementary relations of the coloured FRT algebra

$$R^{-(\lambda,\mu)}L_1^{(\pm)}(\lambda)L_2^{(\pm)}(\mu) = L_2^{(\pm)}(\mu)L_1^{(\pm)}(\lambda)R^{-(\lambda,\mu)} \tag{3.7a}$$

$$R^{-(\lambda,\mu)}L_1^{(-)}(\lambda)L_2^{(+)}(\mu) = L_2^{(+)}(\mu)L_1^{(-)}(\lambda)R^{-(\lambda,\mu)} \tag{3.7b}$$

will also lead to the same quantum algebra $U_q(gl(2))$ and its associated co-product.

4. The Yang–Baxterization of the coloured FRT algebra and construction of the ancestor Lax operator related to the non-additive quantum R-matrix

As mentioned above, a scheme for constructing the single-component spectral-parameter-dependent solution of the QYBE (1.3) through Yang–Baxterization of the FRT algebra (1.1) related to the standard BGR has been proposed in [3]. Such solutions were also applied in constructing Lax operators of a class of quantum integrable models such as the sine–Gordon model, the Liouville model, the Ablowitz–Ladik model and the derivative nonlinear Schrödinger (NLS) model etc, all of which are associated with the additive-type trigonometric R-matrix. Our aim here is to extend this idea to the case of coloured FRT algebra (3.1) related to the CBGR and to explore the possibility of constructing a new class of integrable models associated with a non-additive quantum R-matrix.

The Yang–Baxterization of the CBGR, leading to a two-component spectral-parameter-dependent non-additive R-matrix solution (2.13) of YBE (1.4), was performed in section 2. We would now like to construct solutions of the QYBE (1.3) for the same case

$$R(\lambda,\lambda';\mu,\mu')L_1(\lambda,\lambda')L_2(\mu,\mu') = L_2(\mu,\mu')L_1(\lambda,\lambda')R(\lambda,\lambda';\mu,\mu') \tag{4.1}$$

i.e. to build up both $R(\lambda,\lambda',\mu,\mu')$ and $L(\lambda,\lambda')$ through Yang–Baxterization of the elements $R^{\pm(\lambda,\mu)}$, $L^\pm(\lambda)$ involved in coloured FRT algebra (3.1), (3.7). However, for explicit Yang–Baxterization, we restrict ourselves to $L^{(\pm)}(\lambda)$ (as given by (3.2)) and to the CBGR (as in (2.4), (2.10)), which, in turn, will lead us to the (2×2) Lax operators of physically interesting integrable models with non-additive quantum R-matrices.

In analogy with the Yang–Baxterized R-matrix (2.13) constructed from $R^{\pm(\lambda,\mu)}$ given by (2.4) and (2.10),

$$R(\lambda,\lambda';\mu,\mu') = q^{(\lambda'-\mu')}R^{+(\lambda,\mu)} - q^{-(\lambda'-\mu')}R^{-(\lambda,\mu)} \tag{4.2}$$

we propose now that the form of the associated $L(\lambda,\lambda')$ operator should be

$$L(\lambda,\lambda') = q^{\lambda'}\cdot L^{(+)}(\lambda) + q^{-\lambda'}\cdot L^{(-)}(\lambda) \tag{4.3}$$

where $L^{(\pm)}(\lambda)$ are the upper (lower) triangular matrices given by (3.2*a*), (3.2*b*). Note that the underlying quantum algebra and coalgebra, as shown in section 3, are independent of the colour parameter λ. This essential property allows us to interpret the colour parameters as spectral parameters. Consequently, the L-operator (4.3) may depend on two independent spectral parameters λ and λ': one coming from the colour parameter, the other from the Yang–Baxterization. To verify that the constructed R-matrix (4.2) and L-operator (4.3) are the solutions of the QYBE, we insert them into (4.1) and match the coefficients of different powers in spectral parameters λ', μ'. As a result, we obtain a set of algebraic relations independent of parameters λ', μ' and observe that all of these relations, except one, coincide with the coloured FRT relations (3.1), (3.7) and, hence, are naturally satisfied by construction. The only remaining equation is

$$\begin{aligned} &R^{+(\lambda,\mu)}L_1^{(-)}(\lambda)L_2^{(+)}(\mu) - R^{-(\lambda,\mu)}L_1^{(+)}(\lambda)L_2^{(-)}(\mu) \\ &\qquad = L_2^{(+)}(\mu)L_1^{(-)}(\lambda)R^{+(\lambda,\mu)} - L_2^{(-)}(\mu)L_1^{(+)}(\lambda)R^{-(\lambda,\mu)}. \end{aligned} \tag{4.4}$$

We notice that in the colour-free case ($\lambda = \mu = 0$) relation (4.4) reduces to

$$R^+L_1^{(-)}L_2^{(+)} - R^-L_1^{(+)}L_2^{(-)} = L_2^{(+)}L_1^{(-)}R^+ - L_2^{(-)}L_1^{(+)}R^- \tag{4.5}$$

which was the condition found in [3] as the requirement for Yang–Baxterization of the standard FRT algebra. It was observed further that, satisfying the extra Hecke condition (2.11) for the BGR $R^{\pm}$, equation (4.5) reduced again to some relations of the FRT algebra (1.1). However, in the present case, instead of the Hecke condition (2.11), the colour extension of the BGR $R^{\pm}$

$$R^{+(\lambda,\mu)} - R^{-(\lambda,\mu)} = (q-q^{-1})\cdot \mathcal{P}^{(\lambda,\mu)} \tag{4.6}$$

with $\mathcal{P}^{(\lambda,\mu)} = F^{-1}(\lambda,\mu)\cdot\mathcal{P}\cdot\tilde{F}^{-1}(\lambda,\mu)$ is satisfied, which is obtainable from (2.11) by the symmetry transformation (2.8). We may call $\mathcal{P}^{(\lambda,\mu)}$ the 'coloured permutation operator', which in our concrete case (2.9) may be given by

$$\mathcal{P}^{(\lambda,\mu)} \begin{pmatrix} q^{-(\lambda-\mu)} & & & \\ & 0 & s^{-(\lambda-\mu)} & \\ & s^{\lambda-\mu} & 0 & \\ & & & q^{\lambda-\mu} \end{pmatrix} \tag{4.7}$$

yielding clearly the standard permutation operator $\mathcal{P}$ at $\lambda = \mu$. Now using condition (4.6) and relations of the coloured FRT algebra,.one may further reduce equation (4.4) to

$$\mathcal{P}^{(\lambda,\mu)}\cdot(L_1^{(-)}(\lambda)L_2^{(+)}(\mu) + L_1^{(+)}(\lambda)L_2^{(-)}(\mu)) = (L_2^{(-)}(\mu)L_1^{(+)}(\lambda) + L_2^{(+)}(\mu)L_1^{(-)}(\lambda))\cdot\mathcal{P}^{(\lambda,\mu)}. \tag{4.8}$$

In the colour-free limit this equation is satisfied trivially due to the permuting property of $\mathcal{P}$. However, in this coloured case, the validity of (4.8) is not apparent and should be checked explicitly. In fact, through direct calculation, we find that for the particular form of $L^{(\pm)}(\lambda)$ given by (3.2) and the related permutation operator (4.7), equation (4.8) is indeed satisfied exactly. It is worth mentioning here, that, although the symmetry of the coloured FRT algebra allows independent multiplicative prefactors of $L^{(\pm)}(\lambda)$ as arbitrary functions of Casimir operators (3.4) and colour parameters λ, the Yang–Baxterization restricts the form of $L^{(\pm)}(\lambda)$ to only (3.2*a*), (3.2*b*). Thus, our Yang–Baxterization of the coloured FRT algebra is completed leading to a solution of QYBE (4.1), given by (4.3) and (3.2), as

$$L(\lambda,\lambda')=(\tau_1^-\tau_2^+)^\lambda\begin{pmatrix} q^\Lambda(q^{\lambda'}\tau_1^+ + q^{-\lambda'}\tau_1^-) & s^{-\lambda+\Lambda}q^{-\lambda+\lambda'}\cdot\tau_{21} \\ s^{\lambda-\Lambda}q^{\lambda-\lambda'}\cdot\tau_{12} & q^{-\Lambda}(q^{\lambda'}\tau_2^+ + q^{-\lambda'}\tau_2^-)\end{pmatrix} \tag{4.9}$$

along with the $R(\lambda,\lambda';\mu,\mu')$-matrix (2.15) as constructed through (4.2).

This formal construction of the solutions of the QYBE may now be applied to the theory of integrable systems by interpreting $L(\lambda,\lambda')$ as the corresponding Lax operator associated with the non-additive $R(\lambda,\lambda';\mu,\mu')$-matrix. More precisely, the operator-valued elements occurring in such $L(\lambda,\lambda')$ might be labelled by the index n, corresponding to the nth lattice point, which would give the local Lax operator $L_n(\lambda,\lambda')$. Operators corresponding to different lattice points should act on independent quantum spaces and must commute. This ensures the condition of ultralocality, which is crucial for solving the system through the quantum inverse scattering method (QISM). However, it is curious to observe that, contrary to the usual case, the Lax operator obtained here depends on two spectral parameters λ' and λ. The first one is present in usual cases, while the second one, coming from the colour degrees of freedom, is a special feature of such Lax operators. As a consequence, the transfer matrix $T(\lambda,\lambda')\,\mathrm{tr}\prod_{n=1}^{M}L_n(\lambda,\lambda')$ of these models would also depend on both the spectral parameters and would, in principle, generate two sets of conserved quantities obtained as the expansion coefficients of $\ln T(\lambda,\lambda')$ in λ, and λ' [2]. Although this fascinating feature, in relation to quantum integrable models, might prove to be fruitful, it requires detailed investigation and will not be explored in the present article. Instead, we restrict ourselves to some special cases with single-component spectral parameters.

It is easy to notice that if one sets $\lambda=\mu=0$, $\Lambda=0$, i.e. considers the colour-free limit of $L(\lambda,\lambda')$ (4.12) and $R(\lambda,\lambda';\mu,\mu')$ (2.15), they would reduce respectively to the 'ancestor' Lax operator of [3,4] and the well known additive trigonometric R-matrix related to the six-vertex model. Through different reductions, this 'ancestor' Lax operator was shown [4] to recover a wide class of quantum integrable models including the spin-$\frac{1}{2}$ XXZ chain, the lattice version of the sine–Gordon model [19], the Liouville model and also generated a novel derivative NLS model [20], all of which naturally shared the same trigonometric R-matrix structure. On the other hand, fixing the colour parameters λ,μ as $\lambda=\mu=\theta$, one gets, from (4.9) and (2.15), another 'ancestor' Lax operator with an additive-type R-matrix obtainable from the six-vertex model through some 'gauge transformation' [21] and related to the deformed $\mathrm{GL}_{p,q}(2)$ quantum group [16]. This Lax operator [4] yields another set of integrable models including the $6V(1)$ spin chain [22], the Ablowitz–Ladik model [23] and the relativistic Toda chain [24]. This shows that the construction proposed here, based on the Yang–Baxterization of the coloured FRT algebra, may reproduce different classes of integrable models, considered previously,.at particular limits of the colour parameters.

However, it should be stressed that the construction (equations (4.12) and (2.15)) is powerful enough to yield more interesting classes of models associated with the single-component but non-additive spectral-parameter-dependent R-matrix. To achieve this, we may consider the colour parameters λ, μ not as independent variables, but as some functions of other spectral parameters λ', μ'. For simplicity, we may choose $\lambda = c\lambda' + \theta$, $\mu = c\mu' + \theta$ and set $s = q^{1/c}$, where c and θ are constant parameters. This simple choice reduces the general form of the R-matrix (2.15) to

$$R(\lambda', \mu') = \begin{pmatrix} q^{-c(\lambda'-\mu')a(\lambda'-\mu')} & & & \\ & q^{c(\lambda'+\mu')+2\theta}b(\lambda'-\mu') & \sin\alpha & \\ & \sin\alpha & q^{-c(\lambda'+\mu')-2\theta}b(\lambda'-\mu') & \\ & & & q^{c(\lambda'-\mu')}a(\lambda'-\mu') \end{pmatrix} \tag{4.10}$$

with the same form of $a(\lambda' - \mu')$ and $b(\lambda' - \mu')$ as defined in (2.15). It is remarkable that this R-matrix is evidently non-additive in nature and depends on the sum as well as on the difference of the spectral parameters. It is also easy to observe that, at the $c = 0$ limit, (4.10) reduces to the additive form, recovering the well known case discussed above. The same choice of parameters λ, μ, s and a trivial scaling of the Casimir-like operator Λ, as $\Lambda \to \theta + c\Lambda$, also simplifies the form of the L-operator (4.9) as

$$L(\lambda') = (\tau_1^- \tau_2^+)^{c\lambda'+\theta} \begin{pmatrix} q^{\theta+c\Lambda}(q^{\lambda'} \cdot \tau_1^+ + q^{-\lambda'} \cdot \tau_1^-) & q^{\Lambda-(c\lambda'+\theta)} \cdot \tau_{21} \\ q^{-\Lambda+(c\lambda'+\theta)} \cdot \tau_{12} & q^{-(\theta+c\Lambda)}(q^{\lambda'} \cdot \tau_2^+ + q^{-\lambda'} \cdot \tau_2^-) \end{pmatrix}. \tag{4.11}$$

One of the main results of this section is the explicit form (4.11) for $L(\lambda')$, which may be considered as the ancestor Lax operator of a new class of integrable models satisfying the QYBE (1.3) with a non-additive quantum R-matrix (4.10).

5. Construction of a new class of models through explicit realizations of the ancestor Lax operator

We show here that different realizations of the generators τ of the underlying quantized algebra (3.3), inserted into the ancestor Lax operator (4.11), would generate a specific class of quantum integrable models with a novel non-additive quantum R-matrix (4.10). Let us consider first a realization of generators τ in canonical operators u and p with the standard commutation relation $[u, p] = \mathrm{i}/\Delta$, where Δ is the lattice constant. This may be given as

$$\begin{aligned} &\tau_1^+ = -\tau_2^- = -\tfrac{1}{2}\mathrm{i}m\Delta\mathrm{e}^{\mathrm{i}\alpha u} \qquad \tau_2^+ = -\tau_1^- = -\tfrac{1}{2}\mathrm{i}m\Delta\mathrm{e}^{-\mathrm{i}\alpha u} \\ &\tau_{12} = \mathrm{e}^{-\mathrm{i}\Delta p}g(u) \qquad \tau_{21} = g(u)\mathrm{e}^{\mathrm{i}\Delta p} \end{aligned} \tag{5.1}$$

where $g(u) = [1 + \frac{1}{2}m^2\Delta^2 \cdot \cos 2\alpha(u + \frac{1}{2})]^{1/2}$. Inserting realization (5.1) into the general form (4.14), we obtain the representative Lax operator of an integrable lattice model. Interestingly, at the limit $c = \theta = 0$ and $\Lambda = 0$, one recovers the renowned lattice sine–Gordon (LSG) model [19], associated with the additive R-matrix. Therefore, the present model with non-trivial c, θ and lattice-point-dependent Λ_n which is related to the non-additive quantum R-matrix (4.10) may be considered as a new 'colour' generalization of the LSG model.

Consider now a q-oscillator realization of generators τ in the form

$$\tau_1^+ = \frac{4iq}{\Delta}\tau_2^- = q^{-\mathcal{N}} \qquad \tau_2^+ = -\frac{4i}{\Delta q}\tau_1^- = q^{\mathcal{N}} \qquad \tau_{12} = -\kappa A \qquad \tau_{21} = \kappa A^\dagger \tag{5.2}$$

where $\kappa = ((\Delta/2)\sin 2\alpha)^{1/2}$ and operators $A, A^\dagger, N$ satisfy the q-commutation relations [25, 26]

$$[A, \mathcal{N}] = A \qquad [A^\dagger, \mathcal{N}] = -A^\dagger \qquad AA^\dagger - (\tilde{q})^{\mp 1}A^\dagger A = (\tilde{q})^{\pm\mathcal{N}} \tag{5.3}$$

with $\tilde{q} = q^2 = e^{2i\alpha}$. This realization generates, from the ancestor $L(\lambda')$ (4.11), the Lax operator of a novel integrable model involving q-oscillators, which at the colour-free and $\theta = \Lambda = 0$ limit recovers the lattice version of the derivative NLS model studied in [20]. Therefore, the present model is again a generalization of the lattice derivative NLS model with the inclusion of parameters c, θ and operator Λ_n and is associated with a non-additive quantum R-matrix.

We should observe that the generators τ can have another different realization through q oscillators given in the form

$$\begin{aligned} &\tau_1^+ = \tau_2^- = 0 \qquad \tau_1^- = q^{-(\mathcal{N}+(1/2)+\theta)} \qquad \tau_2^+ = q^{-(\mathcal{N}+(1/2)-\theta)} \\ &\tau_{12} = f(\mathcal{N})A^\dagger \qquad \tau_{21} = Af(\mathcal{N}) \end{aligned} \tag{5.4}$$

where $f^2(\mathcal{N}) = (q^{-1} - q)q^{-\mathcal{N}}$ and operators $A, A^\dagger, N$ satisfy the same commutation relation (5.3) but with $\tilde{q} = q$. When inserted into $L(\lambda')$ (4.11), this realization yields yet another quantum integrable lattice model depending on external parameters θ, c and lattice-point-dependent operator Λ_n. To see the physical relevance of this model, we fix the value of parameter θ as $-\frac{1}{2}$, which greatly simplifies the representative Lax operator to yield

$$L_n(\lambda') = q^{-2(N_n+(1/2))c\lambda'} \begin{pmatrix} q^{-\lambda'+c\Lambda_n} & q^{(\Lambda_n - c\lambda')} \cdot b_n \\ q^{(c\lambda'-\Lambda_n)} \cdot b_n^\dagger & q^{\lambda'-c\Lambda_n} \end{pmatrix} \tag{5.5}$$

where $b \equiv Aq^{\mathcal{N}/2}(q^{-1} - q)^{1/2}$ and $b^\dagger \equiv q^{\mathcal{N}/2}(q^{-1} - q)^{1/2}A^\dagger$ is another form of the q-oscillator with algebra $q^{-2}bb^\dagger - b^\dagger b = q^{-2} - 1$ as introduced by Macfarlane [25]. Notice that, at the limit $c = \Lambda_n = 0$, the Lax operator (5.5) coincides exactly with that of the well known Ablowitz–Ladik model [23]. Consequently, the model represented by the Lax operator (5.5) is a quantum lattice model associated with the non-additive R-matrix (4.10) (with $\theta = -\frac{1}{2}$), and is an integrable generalization of the Ablowitz–Ladik model.

In a similar way, starting from Lax operator (4.11) and taking other realizations of the generators τ, one may expect to construct other different quantum integrable lattice models, including generalizations of the lattice Liouville model, the relativistic Toda chain etc. It should be emphasized again that all these varieties of integrable models belong to the same descendent class since they are obtainable from the same ancestor model (4.11) and share the same non-additive quantum R-matrix (4.10), which itself is a new result. The construction of the Hamiltonian and the determination of the energy spectrum of this class of models through the QISM deserves detailed investigation and will not be considered here.

6. Conclusion

The FRT algebra related to the quantum group is able to generate a wide class of quantum integrable models through Yang–Baxterization. However, such models, which also include the sine–Gordon model, the spin-$\frac{1}{2}$ XXZ chain, the Ablowitz–Ladik model and the derivative NLS model etc, are all associated with additive-type R-matrices. These R-matrices, in turn, are obtainable again from the standard BGR through Yang–Baxterization.

On the other hand, there exist physical models with non-additive R-matrices. For example, the one-dimensional Hubbard model is related to a quantum R-matrix which depends on the sum, as well as the difference, of the spectral parameters. Similarly, a ferromagnetic model in external fields also exhibits non-additive-type dependence on the R-matrix. With the aim of generating such classes of integrable models, we focus here on the colour realization of the FRT algebra and succeed in constructing a new 'ancestor' Lax operator through its Yang–Baxterization. The corresponding non-additive-type R-matrix is also obtained by starting from the coloured BGR.

Interestingly, together with this promising scheme we are also able to find explicit forms of the elements $L^{(\pm)}(\lambda)$ for the coloured FRT algebra in the $U_q(gl(2))$ case. Though these (upper or lower triangular) matrices are found to depend manifestly on the colour parameters, the underlying quantum algebra and the associated co-product determined by them turn out to be the standard ones, devoid of any colour parameters. To achieve our goal of generating integrable models, we have had to further Yang–Baxterize this coloured algebra to find some algebraic relations for the consistency condition. In the standard case, the extra Hecke condition itself trivially satisfies such relations. In the coloured case, however, the situation become more complicated and their validity has to be checked by direct computation.

It is important to stress that the Lax operator and the R-matrix, found through such Yang–Baxterization of the coloured FRT algebra, represent, in general, an altogether new class of models with two-component spectral parameters. One of these spectral parameters comes from the colour degrees of freedom, while the other one is the usual parameter introduced through Yang–Baxterization. Such models might be of physical interest and should be investigated separately. At the colour-free case, as expected, we recover the large variety of models obtained previously by us [4,19], while for the simplest choice of linear dependence between two sets of spectral parameters a new class of quantum integrable models emerges, which is associated with an interesting non-additive quantum R-matrix.

Different realizations of the same underlying quantized algebra in more physical entries, like bosons or q-oscillators, are found to yield a variety of quantum integrable systems; for example, the 'coloured' extensions of the lattice sine–Gordon model, the Ablowitz–Ladik model or a lattice derivative NLS model etc. It is curious to note that the associated R-matrix of all such models turns out to be dependent only on the sum and difference of the spectral parameters, just as in the case of the one-dimensional Hubbard model. The problem of constructing the Hamiltonians of these models, along with the energy spectrum through the QISM, has not been carried out here and should be studied separately. It is also worth investigating whether the class of lattice models obtained here would also generate new integrable field models at the continuum limit, in parallel to the colour-free case. The extension of the scheme presented here to semisimple algebras would also be a promising problem.

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